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An Extension of Waring's Problem

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PHILOSOPHICAL TRANSACTIONS.

I. *An Extension of Waring's Problem.*

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Introduction.

1.1. An interesting extension of WARING'S famous problem is the following:—

Can every sufficiently large n be expressed as the sum of s *almost equal* k-th powers; or, more generally, can every sufficiently large n be expressed as the sum of s positive k-th powers *almost proportional* to s arbitrarily assigned positive numbers $\lambda_1, \lambda_2, \ldots \lambda_s$?

I have developed two methods to discuss this problem, one based on the Hardy-Littlewood method for the solution of WARING'S problem and the other on the new VINOGRADOFF method* for the solution of the same problem. In this paper I shall discuss the case $k \geq 3$ by the first of these methods. The case of five or more squares may be treated in the same way, and the results are similar; somewhat deeper and more troublesome analysis is required to deal with the case of four squares.

The principle of the method employed here is that of "weighting" the various representations of n as the sum of s k-th powers in such a way as to make predominant the particular representation of which we are in search.

Two theorems in this direction have been proved by CHOWLA† by "elementary" methods. They are as follows:—

Given $\varepsilon > 0$, *every odd* $n > n_0(\varepsilon)$ *can be expressed in the form*

$$n = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

with every $m_i > n^{\frac{1}{2}-\epsilon}$.

Every large integer can be expressed in the form

$$n = m_1^3 + m_2^3 + \ldots + m_8^3$$

with every m_i *greater than* $\frac{1}{9} n^{1/3}$.

These results are not included in those of the present paper. I have succeeded in finding a result which includes the first, but I am unable to prove the second by either of my methods.

* GELBKE, 'Math. Ann.,' vol. 105, p. 637 (1931).

† 'J. Indian Math. Soc.,' vol. 18, p. 129 (1929), and 'J. Math Soc. Lond.,' vol. 5, p. 155 (1930).

1.2. The argument involves a number of parameters and of letters used in a conventional sense, so that the system of notation requires careful explanation.

k, s, n, m_i, $(i = 1, 2, \ldots, s)$ are positive integers, $k > 2$ and $K = 2^{k-1}$. h, l, m are positive integers or zero. We write

$$s_1 = (k-2)K + 5, \qquad s_2 = (\tfrac{1}{2}k - 1)K + 3.$$

γ, λ_1, $\lambda_2, \ldots \lambda_s$ are arbitrary positive numbers. λ will be used to denote the whole set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$; that is $l = l(\lambda)$ means that l depends on the s parameters $\lambda_1, \lambda_2, \ldots, \lambda_s$. We note that $l = l(\lambda)$ will therefore imply that l depends on s. We write

$$\Lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_s.$$

If n can be expressed in the form

$$n = m_1^k + m_2^k + \ldots + m_s^k, \qquad (1.21)$$

where

$$(1-\gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i^k}{n} < (1+\gamma)\frac{\lambda_i}{\Lambda} \quad (i = 1, 2, \ldots s), \qquad (1.22)$$

then we shall say that n is the sum of s k-th powers proportional "within γ" to λ_1, $\lambda_2, \ldots, \lambda_s$.

The principal variables are n, k, s, γ, λ, l, γ', ε and δ. Of these γ, γ' and ε are arbitrary positive numbers, not always the same; δ is also positive. γ, γ', ε and δ are to be thought of as small. The choice of δ, which has to be made to satisfy the varying requirements of the analysis, is always subsequent to that of k, s and ε.

The letters A, B, C, D, with or without suffixes, denote a positive number whose value varies from one occurrence to another. When no suffix is used, A is an absolute constant (such as 3); $B = B(k, l)$, a function of k and l only; $C = C(k, \lambda)$, a function of k, $\lambda_1, \lambda_2, \ldots \lambda_s$, (and therefore of s) only; $D = D(k, \lambda, l)$. Sometimes, however, A, B, C, D will depend on other parameters, in which case these parameters will be indicated explicitly by suffixes. Thus,

$$B_\varepsilon = B(k, l, \varepsilon), \qquad C_{\varepsilon, \delta} = C(k, \lambda, \varepsilon, \delta).$$

In exponents c is used to denote a positive function of k and s only, whose value varies from one occurrence to another. When c occurs not in an exponent, our conventions do not apply, and all the variables on which c depends are shown by suffixes.

We shall quote certain known results by reference to LANDAU.*

1.3. The principal results of this paper are as follows :—

Theorem 1.—*If* $s \geq s_1$, *then for any positive values of* γ, $\lambda_1, \ldots, \lambda_s$, *there exists a number* $n_0 = n_0(k, \gamma, \lambda)$ *such that every integer greater than* n_0 *is the sum of* s *positive* k*-th powers proportional "within* γ*" to* $\lambda_1, \lambda_2, \ldots, \lambda_s$.

* "*Vorlesungen über Zahlentheorie*," vol. I.

Theorem 2.—*If* $k \neq 4$, $s \geq s_2$, *and* $\gamma, \lambda_1, \ldots \lambda_s$, *are any positive numbers, almost all positive integers are the sum of s positive k-th powers proportional "within* γ*" to* $\lambda_1, \lambda_2, \ldots, \lambda_s$. *The number of integers less than n for which this is not true is less than* $C_\gamma n^{1-c}$.

Theorem 3.—*If* $s \geq 15$, *and* $\gamma, \lambda_1, \ldots, \lambda_s$, *are any positive numbers, almost all positive integers are the sums of s positive fourth powers proportional "within* γ*" to* $\lambda_1, \lambda_2, \ldots \lambda_s$. *The number of integers less than n for which this is not true is less than* $C_\gamma n^{1-c}$.

These results correspond to those of HARDY and LITTLEWOOD* :—

1. *Every sufficiently large n is the sum of* s_1 *k-th powers.*

2. *Almost all integers are the sums of* s_2 *k-th powers* ($k \neq 4$). *The number of integers not so representable and less than n is less than* Cn^{1-c}.

3. *Almost all integers are the sums of* 15 *fourth powers.*

These authors improve their results considerably by assuming the truth of the unproved "hypothesis K"†:—"*The number of representations of n as the sum of k k-th powers is less than* Bn^ϵ *for every positive* ϵ." On the same assumption, we can prove the corresponding improved forms of my Theorems 1 and 2.

1.4. We first show that it is sufficient to prove these three theorems for the particular case in which the numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$ are arbitrary positive integers.

It is obvious from (1.22) that we are only concerned with the numbers

$$\frac{\lambda_1}{\Lambda}, \frac{\lambda_2}{\Lambda}, \ldots, \frac{\lambda_s}{\Lambda}, \qquad (1.41)$$

and so we can give Λ any positive value we choose without loss of generality. Let us first take $\Lambda = 1$. Then, if any λ_i is irrational, at least one other must be irrational, and so, for any $\gamma > 0$, we can choose positive rational numbers $\lambda_1', \lambda_2', \ldots, \lambda'_s$, depending only on $\gamma, \lambda_1, \ldots, \lambda_s$, such that

$$\lambda_1' + \lambda_2' + \ldots + \lambda_s' = 1$$

and that $|\lambda_i - \lambda_i'|$ is so small that

$$(1-\gamma)\,\lambda_i < (1-\tfrac{1}{2}\gamma)\,\lambda_i' < (1+\tfrac{1}{2}\gamma)\,\lambda_i' < (1+\gamma)\,\lambda_i, \quad (i = 1, 2, \ldots, s).$$

Then, if any number is the sum of s k-th powers proportional "within $\tfrac{1}{2}\gamma$" to λ_1', $\lambda_2', \ldots, \lambda_s'$, these powers are also proportional "within γ" to $\lambda_1, \lambda_2, \ldots, \lambda_s$. Also, in Theorem 1,

$$n_0 = n_0\,(k, \tfrac{1}{2}\gamma, \lambda') = n_0\,(k, \gamma, \lambda),$$

* LANDAU, p. 239, or the original memoirs of HARDY and LITTLEWOOD, 'Math. Z.,' vol. 12, p. 161 (1922), and vol. 23, p. 1 (1925). In 1, if $k > 3$, s_1 can be replaced by a smaller number. The method of obtaining this final step does not appear capable of extension to the problem of this paper.

† See the second memoir referred to above.

since $\lambda' = \lambda'(\gamma, \lambda)$. Similarly, in Theorems 2 and 3, we have

$$C_{\frac{1}{2}\gamma} = C(k, \tfrac{1}{2}\gamma, \lambda') = C(k, \gamma, \lambda).$$

Then it is sufficient to prove the theorems for the case in which every λ_i is rational.

We see then that we must prove the theorems for all positive values of γ and all positive rational values of the numbers (1.41). But for this purpose it is obviously sufficient to prove the theorems for all positive values of γ and all positive integral values of $\lambda_1, \lambda_2, \ldots, \lambda_s$. Henceforth, then, λ_i denotes an arbitrary positive integer.

Further Definitions and Theorems.

2.1. We use Π to denote $\prod_{i=1}^{s}$; also $\prod_{i\neq i'}$ means that i takes all values from 1 to s except i'. $m_1, m_2, \ldots, m_s$ are any set of positive integers satisfying condition (1.21) for a given n, if such a set exists;

$$m_i^k = \frac{\lambda_i}{\Lambda}\alpha_i n, \quad P(n) = P(n, m_1, m_2, \ldots, m_s, \lambda) = \Pi(\alpha_i^{\lambda_i}),$$

and

$$\bar{P}(n) = \bar{P}(n, k, \lambda) = \operatorname{Max} P(n, m_1, m_2, \ldots, m_s, \lambda)$$

under the same condition. $\bar{P}(n)$ is defined for a particular n if, and only if, there is at least one solution of (1.21) in positive integers.

Consider the Λ numbers

$$\alpha_1, \alpha_1, \ldots (\lambda_1 \text{ times}), \alpha_2, \alpha_2, \ldots (\lambda_2 \text{ times}), \alpha_3, \alpha_3, \ldots (\lambda_3 \text{ times}), \alpha_4, \ldots . \quad (2.11)$$

Their sum is

$$\sum_{i=1}^{s} \lambda_i\alpha_i = \frac{\Lambda}{n}\sum_{i=1}^{s} m_i^k = \Lambda;$$

and so the arithmetic mean is unity. Hence, their geometric mean is less than or equal to unity, and so is the product, that is

$$\Pi\alpha_i^{\lambda_i} = P(n) \leq 1.$$

Again, n is expressible as the sum of s k-th powers proportional "within γ" to $\lambda_1, \lambda_2, \ldots, \lambda_s$ if, and only if, at least one set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ exists, and satisfies the condition

$$1 - \gamma < \alpha_i < 1 + \gamma, \quad (i = 1, 2, \ldots, s).$$

Then we have

$$(1-\gamma)^{\Lambda} < P(n) \leq 1.$$

We shall see later (Lemma 24) that, conversely, if

$$P(n) > \left(1 - \left(\frac{\gamma}{2\Lambda}\right)^2\right)^{\Lambda},$$

then

$$1 - \gamma < \alpha_i < 1 + \gamma, \quad (i = 1, 2, \ldots, s).$$

We can take $P(n) = \bar{P}(n)$ for this purpose, since we need only to prove that one set of α_i fulfils the conditions.

These considerations lead us to formulate three further theorems, from which we shall finally deduce Theorems 1, 2 and 3.

Theorem 4.—*If* $s \geq s_1$, *and* $\gamma' > 0$, *then* $\bar{P}(n)$ *is defined and*

$$\bar{P}(n) > 1 - \gamma',$$

for $n > n_0 = n_0(k, \lambda, \gamma')$.

Theorem 5.—*If* $s \geq s_2$, $k \neq 4$, *and* $\gamma' > 0$, *then, for almost all* m, $\bar{P}(m)$ *is defined and*

$$\bar{P}(m) > 1 - \gamma'.$$

The number of integers less than n for which this is not true is less than $C_{\gamma'} n^{1-c}$.

Theorem 6.—*If* $k = 4$ *and* $s \geq 15$, *the conclusions of Theorem* 5 *still hold good.*

2.2. We now introduce certain new functions to enable us to study the behaviour of $\bar{P}(n)$. We take $|x| < 1$ and l a positive integer or zero, and write

$$f = f(x) = \sum_{h=1}^{\infty} h^{lk} x^{h^k}, \quad f_i = f_i(x) = \sum_{h=1}^{\infty} h^{\lambda_i lk} x^{h^k}, \quad \Pi f_i = \sum_{n=1}^{\infty} r_l(n)\, x^n.$$

$\sum_{(n)}$ denotes summation over all positive integral values of $m_1, m_2, \ldots, m_s$ which satisfy (1.21). Then we see that

$$r_0(n) = \sum_{(n)} 1,$$

$$r_l(n) = \sum_{(n)} (\Pi m_i^{\lambda_i lk}) = n^{\Lambda l} \Lambda^{-\Lambda l} \Pi \lambda_i^{\lambda_i l} \sum_{(n)} (P(n))^l \quad \ldots\ldots \quad (2.21)$$

$$\leq (n^{\Lambda} \bar{P}(n) \Lambda^{-\Lambda} \Pi \lambda_i^{\lambda_i})^l r_0(n);$$

also, if $r_0(n) > 0$, then $\bar{P}(n)$ is defined and we have

$$\bar{P}(n) \geq \frac{\Lambda^{\Lambda}}{n^{\Lambda} \Pi \lambda_i^{\lambda_i}} \left(\frac{r_l(n)}{r_0(n)}\right)^{1/l} \quad \ldots\ldots \quad (2.22)$$

2.3. We see then that we must study the behaviour of $r_l(n)$ for $l \geq 0$. For this purpose we require certain further symbols.

q is a positive integer, p is a positive integer less than and prime to q, unless $q = 1$, when $p = 0$ only. $\sum_p$ denotes summation over all values of p associated with a particular q.

We also write

$$a = \frac{1}{k}, \quad \kappa = 1 - \frac{1}{K}, \quad \Delta_l = \frac{a^s \Pi \Gamma (\lambda_i l + a)}{\Gamma (\Lambda l + sa)},$$

$$e(x) = e^{2\pi i x}, \quad e_q(x) = e\left(\frac{x}{q}\right), \quad S_{p,q} = \sum_{h=0}^{q-1} e_q(ph^k),$$

$$S(n) = \sum_{q=1}^{\infty} \sum_{p} \left(\frac{S_{p,q}}{q}\right)^s e_q(-np),$$

$$\rho_l(n) = \Delta_l n^{\Lambda l + sa - 1} S(n), \quad \sigma_l(n) = r_l(n) - \rho_l(n).$$

$[v]$ denotes the integral part of v.

$b = a - \delta$, where δ is a function of k, s and ε tending to zero with ε. δ will be fixed later.

Γ is the circle

$$|x| = e^{\ 1/n}$$

in the complex x-plane. On Γ I write

$$x = |x|\, e^{i\psi} = e^{-\frac{1}{n} + i\psi}.$$

We take the Farey-dissection of order

$$N = [n^{1-b}]$$

on the circumference of the circle Γ. Then, if $1 \leq q \leq N$, we have an arc $\xi_{p,q}$ associated with every point.

$$x = \exp\left(-\frac{1}{n} + \frac{2p\pi i}{q}\right).$$

In connection with the arc $\xi_{p,q}$, we write

$$x = e_q(p)\, X = e_q(p)\, e^{-y}, \quad y = \frac{1}{n} - i\theta, \quad \theta = \psi - \frac{2p\pi}{q},$$

where that value of ψ is taken which makes $-\pi \leq \theta \leq \pi$. Then we know that, on $\xi_{p,q}$, θ varies between two numbers $-\theta'_{p,q}$ and $\theta''_{p,q}$ such that

$$\frac{\pi}{qN} \leq \theta'_{p,q} < \frac{2\pi}{qN}, \quad \frac{\pi}{qN} \leq \theta''_{p,q} < \frac{2\pi}{qN}.$$

$\xi_{p,q}$ will be called a major arc if $q \leq n^b$, and a minor arc if $q > n^b$. A major arc will be denoted by $\mathbf{M}$ or $\mathbf{M}_{p,q}$; a minor arc by $\mathbf{m}$ or $\mathbf{m}_{p,q}$. $\overline{\mathbf{M}}_{p,q}$ is the complementary arc of $\mathbf{M}_{p,q}$ on Γ, that is

$$\mathbf{M}_{p,q} + \overline{\mathbf{M}}_{p,q} = \Gamma.$$

We write also

$$\phi = \phi_{p,q} = \frac{S_{p,q}}{q} \frac{a\Gamma(l+a)}{y^{l+a}}, \quad \phi_i = \phi_{p,q,i} = \frac{S_{p,q}}{q} \frac{a\Gamma(\lambda_i l + a)}{y^{\lambda_i l + a}};$$

$$g(z) = \sum_{m=1}^{\infty} m^{\Delta l + sa - 1} z^m, \quad (|z| < 1),$$

$$F(x) = F(x, k, l, \lambda) = \sum_{m=1}^{\infty} \rho_l(m) x^m$$

$$= \Delta_l \sum_{m=1}^{\infty} S(m) m^{\Delta l + sa - 1} x^m,$$

and

$$F_{p,q} = \Delta_l \left(\frac{S_{p,q}}{q}\right)^s g(xe_q(-p)).$$

Finally, we write

$$S_1(n) = \sum_{q \le \nu} \sum_p \left(\frac{S_{p,q}}{q}\right)^s e_q(-np),$$

$$S_2(n) = \sum_{q > \nu} \sum_p \left(\frac{S_{p,q}}{q}\right)^s e_q(-np),$$

so that

$$S(n) = S_1(n) + S_2(n);$$

and

$$F_1(x) = \Delta_l \sum_{m=1}^{\infty} m^{\Delta l + sa - 1} S_1(m) x^m, \quad F_2(x) = \Delta_l \sum_{m=1}^{\infty} m^{\Delta l + sa - 1} S_2(m) x^m,$$

so that

$$F(x) = F_1(x) + F_2(x).$$

Then,

$$F_1(x) = \sum_{q \le \nu} \sum_p F_{p,q}, \quad F_2(x) = \sum_{q > \nu} \sum_p F_{p,q}.$$

We shall take

$$\nu = n^\beta;$$

β will be chosen later.

We suppose always that $n > 1$.

2.4. Our first object is to obtain upper bounds for

$$|\sigma_l(n)| \text{ and } \sum_{m=1}^{n} (\sigma_l(m))^2.$$

Theorem 7.—*For* $s \ge s_1$, *we have*

$$|\sigma_l(n)| < Dn^{\Delta l + sa - 1 - c}.$$

Theorem 8.—*For* $s \ge s_2$, we have

$$\sum_{m=1}^{n} (\sigma_l(m))^2 < Dn^{2\Delta l + 2sa - 1 - c}.$$

The proofs of these last two theorems depend on a series of lemmas. Our method consists in showing that we have results for $f(x)$ and Πf_i for general l, similar to those proved by HARDY and LITTLEWOOD* for the case $l = 0$.

Trivial and Known Results.

3.1. *Lemma* 1.—If $|z| < 1$, $\alpha > 0$, *and*

$$-2\pi + \alpha < \mathfrak{I}(\log z) = \arg z < 2\pi - \alpha,$$

then

$$|g(z)| < D_\alpha |\arg z|^{-\Lambda l - s a}$$

and

$$|g(z) - \Gamma(\Lambda l + sa)(\log 1/z)^{-sa}| < D_\alpha. \quad (3.11)$$

These are known results.

Lemma 2.—*We have*

$$\Sigma \int_{\xi_{p,q}} |F_{p,q} - \Pi \phi_{p,q,i}| \, d\theta < D; \quad (3.12)$$

and

$$\Sigma \int_{\xi_{p,q}} |F_{p,q} - \Pi \phi_{p,q,i}|^2 \, d\theta < D. \quad (3.13)$$

These are trivial; if $z = xe_q(-p) = e^{-y}$, then $y = \log(1/z)$, so that we have

$$|F_{p,q} - \Pi \phi_i| = \left| \Delta_l \left(\frac{S_{p,q}}{q}\right)^s (g(xe_q(-p)) - \Gamma(\Lambda l + sa)\, y^{-\Lambda l - sa}) \right| < D$$

by suitable choice of α in Lemma 1.

Lemma 3.†—*We have*

$$\left|\frac{S_{p,q}}{q}\right| < A_k q^{-a}. \quad (3.14)$$

Lemma 4.—*We have*

$$|S_2(m)| < C\nu^{2-sa}, \quad \text{for} \quad s \geq s_1; \quad (3.15)$$

$$|S_2(m)| < C\nu^{-c} m^{\epsilon}, \quad \text{for} \quad s \geq s_2. \quad (3.16)$$

If $s \geq 2k + 1$,

$$|S_2(m)| \leq \sum_{q>\nu} \sum_p \left|\frac{S_{p,q}}{q}\right|^s < C \sum_{q>\nu} q^{1-sa} < C\nu^{2-sa}.$$

* When $l = 0$, Πf_i becomes $\left(\sum_{h=1}^{\infty} x^{h^k}\right)^s$. HARDY and LITTLEWOOD use $1 + 2\sum_{h=1}^{\infty} x^{h^k}$ in place of $\sum_{h=1}^{\infty} x^{h^k}$. The distinction is unimportant.

† LANDAU, Satz 315.

If $k \geq 3$ and $s \geq s_1$, then $s \geq 2k + 1$. Hence, (3.15) is proved.
If $k \geq 4$ and $s \geq s_2$, then $s \geq 2k + 1$. Hence,

$$| S_2 (m) | < C\nu^{2-sa} \leq C\nu^{-a} < C_\epsilon \nu^{-a} m^\epsilon.$$

If $k = 3$, then $s_2 = 5$, and (3.16) is a known result.*

The Major Arcs.

4.1. *Lemma* 5.—*On a major arc* $M_{p,q}$, *we have*

$$|f - \phi_{p,q}| < B_{\epsilon,\delta}\, q^{lk+\kappa+\epsilon},$$

so that

$$|f - \phi_{p,q}| < B_\epsilon q^{lk+\kappa+\epsilon};$$

if δ *be chosen appropriately as a function of* k *and* ϵ.

Lemmas 6–10 are needed for the proof of Lemma 5.

Lemma 6.†—*If* μ *is a positive integer or zero, then*

$$\left| \sum_{h=0}^{q-1} h^\mu e_q (ph^k) \right| < A_{k,\epsilon} q^{\mu+\kappa+\epsilon}.$$

4.2. *Lemma* 7.—*Suppose that*

$$0 \leq h < q, \quad Y = q^k y, \quad t = \frac{h}{q},$$

$$V = -\frac{d}{dv}\left\{(v+t)^{lk} e^{-Y(v+t)^k}\right\},$$

$$f_{(h)} = \sum_{m=0}^{\infty} (mq + h)^{lk} e^{-(mq+h)^k y} = \sum_{m=0}^{\infty} \tau_m{}^l e^{-\tau_m y},$$

and that $R(v)$, $R_1(v)$, $R_2(v)$, . . . *are polynomials in* $(v) = v - [v]$, *defined successively by*

$$\bar{R}(v) = v + 1 - (v) = v + R(v) = c_0 v + R(v)$$

$$\bar{R}_1(v) = \int_0^v R(w)\, dw = c_1 v + R_1(v),$$

$$\bar{R}_2(v) = \int_0^v R_1(w)\, dw = c_2 v + R_2(v),$$

so that

$$R(v) = 1 - (v),\ R_1(v) = \tfrac{1}{2}(v) - \tfrac{1}{2}(v)^2,\ c_0 = 1,\ c_1 = \tfrac{1}{2},$$

* LANDAU, Satz 327.

† LANDAU, Satz 328.

and in general c_ν is a function of ν only. Then*

$$\frac{f_{(h)}}{q^{kl}} = \sum_{\mu=0}^{\nu} (-1)^\mu c_\mu \int_0^\infty v\, \mathrm{V}^{(\mu)}\, dv + (-1)^\nu \mathrm{R}_\mu (v)\, \mathrm{V}^{(\nu)}\, dv$$

$$= \sum_{\mu=0}^{\nu} (-1)^\mu c_\mu \mathrm{I}_\mu + \mathrm{J}_\nu. \quad \text{(4.21)}$$

If we define N (x) by

$$\mathrm{N}(x) = m+1, \quad (\tau_m \leq x \leq \tau_{m+1}),$$

we have

$$f_{(h)} = \sum_{m=0}^{\infty} \tau_m{}^l e^{-\tau_m y} = -\int_{\mathrm{T}_0}^\infty \mathrm{N}(u) \frac{d}{du}(u^l e^{-uy})\, du.$$

Writing $u = (vq+h)^k = q^k (v+t)^k$, and observing that $\mathrm{N}(u) = \overline{\mathrm{R}}(v)$, we obtain

$$\frac{f_{(h)}}{q^{kl}} = \int_0^\infty \overline{\mathrm{R}}(v)\, \mathrm{V} dv = \int_0^\infty v\mathrm{V}\, dv + \int_0^\infty \overline{\mathrm{R}}_1{}'(v)\, \mathrm{V} dv$$

$$= \int_0^\infty v\mathrm{V}\, dv - c_1 \int_0^\infty v\mathrm{V}'\, dv - \int_0^\infty \mathrm{R}_1(v)\, \mathrm{V}'\, dv.$$

This is (4.21) with $\nu = 1$, and it is plain that the general formula follows from a repetition of the argument.

Lemma 8.—*If*

$$\mathrm{H} = \mathfrak{R}(\mathrm{Y}) = \mathfrak{R}(q^k y) = \frac{q^k}{n},$$

then

$$|\mathrm{Y}| < \frac{|\mathrm{Y}|}{\mathrm{H}^{1-a}} < \mathrm{A}n^{-\delta} \quad \text{(4.22)}$$

on $\mathrm{M}_{p,q}$.

For we have $\mathrm{H} \leq n^{bk-1} < 1$, and

$$\frac{|\mathrm{Y}|}{\mathrm{H}^{1-a}} = qn^{1-a}\,|\,y\,| < \mathrm{A}n^{b-a} = \mathrm{A}n^{-\delta}.$$

4.3. *Lemma* 9.—*We can choose* $\nu_0 = \nu_0(k, l, \lambda)$, *so that*

$$|\mathrm{J}_\nu| = \left|\int_0^\infty \mathrm{R}_\nu \mathrm{V}^{(\nu)}\, dv\right| < \frac{\mathrm{B}_\delta}{n}. \quad \text{(4.31)}$$

We take $\nu = k(\nu_1 + l)$, where $\nu_1 > 2$. We have

$$\mathrm{V}^{(\nu)} = -\left(\frac{d}{dv}\right)^{\nu+1} \{(v+t)^{lk} e^{-\mathrm{Y}(v+t)^k}\}$$

$$= e^{-\mathrm{Y}(v+t)^k} \sum_{j,\,m} c_{k,\,l,\,j,\,m} \mathrm{Y}^j (v+t)^m.$$

* Throughout section 4, ν is an integer, and is quite distinct from $\nu = n^\beta$ of the remainder of the paper.

Every differentiation of the exponential factor of V introduces a factor Y, and a term in Y^j can be generated only by j such differentiations together with $\nu + 1 - j$ which do not bear upon the exponential. Hence the term in Y^j is a multiple of

$$\left(\frac{d}{dv}\right)^{\nu+1-j} \{(-kY)^j (v+t)^{lk+j(k-1)}\},$$

and is zero unless

$$lk + j(k-1) \geq \nu + 1 - j,$$

or

$$j \geq \frac{\nu+1}{k} - l > \nu_1.$$

That is to say, $\nu_1 + 1$ is the least possible value of j. Associated with a given j we have

$$m = lk + j(k-1) - \nu - 1 + j = k(j - \nu_1) - 1.$$

It follows that

$$|V^{(\nu)}| \leq B_\nu e^{-H(v+t)^k} \sum_{j=\nu_1+1}^{\nu} |Y|^j |v+t|^{k(j-\nu_1)-1},$$

and so, using (4.22), we have

$$\begin{aligned}\left|\int_0^\infty R_\nu V^{(\nu)} dv\right| &< B_\nu \sum_{j=\nu_1+1}^{\nu} |Y|^j \int_0^\infty e^{-H(v+t)^k} |v+t|^{k(j-\nu_1)-1} dv \\ &< B_\nu \sum_{j=\nu_1+1}^{\nu} |Y|^j H^{-(j-\nu_1)} \\ &= B_\nu \sum_{j=\nu_1+1}^{\nu} |Y|^j H^{-j(1-a)-aj+\nu_1} \\ &< B_\nu \left(\frac{n}{q^k}\right)^{a\nu-\nu_1} \sum_{j=\nu_1+1}^{\nu} \left(\frac{|Y|}{H^{1-a}}\right)^j \\ &< B_{\nu,\delta} \left(\frac{n}{q^k}\right)^l n^{-(\nu_1+1)\delta} < \frac{B_\delta}{n},\end{aligned}$$

if $\nu_1 = \nu_{1,0}(k, l, \delta)$, that is, if $\nu = \nu_0(k, l, \delta)$.

It should be observed that we can always replace $\nu_0(k, l, \delta)$ by a larger number of the same type.

4.4. We understand by $Q(Y, t)$ a polynomial in Y and t, whose degree in either variable has an upper bound of the type B_δ, and the moduli of whose coefficients have upper bounds of the same type. Since $|Y| < A$, $0 \leq t < 1$, we have always

$$|Q(Y, t)| < B_\delta.$$

And we understand by E a number whose absolute value is less than B_δ/n.

Lemma 10.—*We can choose* $\nu_0 = \nu_0(k, l, \delta)$ *so that in addition to* (4.31), *we have*

$$I_\mu = \int_0^\infty v V^{(\mu)} dv = Q(Y, t) + E, \quad \ldots \ldots \ldots \quad (4.41)$$

for $1 \leq \mu \leq \nu = \nu_0$, *and*

$$I_0 = \int_0^\infty v V \, dv = a \Gamma (l + a) Y^{-l-a} + Q (Y, t) + E. \quad . \quad . \quad . \quad . \quad (4.42)$$

We have

$$e^{-Yt^k} = \sum_{m=0}^{m'} \frac{(-Yt^k)^m}{m!} + r,$$

where

$$|r| < A |Y|^{m'} < A (An^{-\delta})^{m'} < \frac{B_\delta}{n},$$

if $m' = m'(k, \delta)$ is sufficiently large; and so

$$e^{-Yt^k} = Q (Y, t) + E.$$

If $\mu > 1$, we have

$$I_\mu = \int_0^\infty v V^{(\mu)} \, dv = V^{(\mu-2)} (0) = e^{-Yt^k} Q (Y, t)$$

$$= Q (Y, t) \{Q (Y, t) + E\} = Q (Y, t) + E;$$

while, if $\mu = 1$,

$$I_1 = \int_0^\infty v V^{(1)} \, dv = -\int_0^\infty V \, dv$$

$$= \int_0^\infty \frac{d}{dv} \{(v + t)^{lk} e^{-Y(v+t)^k}\} \, dv$$

$$= -t^{lk} e^{-Yt^k} = Q (Y, t) + E.$$

Finally, if $\mu = 0$, we have

$$I_0 = \int_0^\infty v V \, dv = \int_0^\infty (v + t)^{lk} e^{-Y(v+t)^k} \, dv$$

$$= \int_t^\infty w^{lk} e^{-Yw^k} \, dw$$

$$= a \Gamma (l + a) Y^{-l-a} - \int_0^t w^{lk} e^{-Yw^k} \, dw$$

$$= a \Gamma (l + a) Y^{-l-a} - \int_0^t (Q (Y, w) + E) \, dw$$

$$= a \Gamma (l + a) Y^{-l-a} + Q (Y, t) + E.$$

4.5. We can now prove Lemma 5. We have

$$f = -1 + \sum_{h=0}^{q-1} e_q (ph^k) \sum_{m=0}^{\infty} \tau_m{}^l e^{-\tau_m y}$$

$$= -1 + \sum_{h=0}^{q-1} e_q (ph^k) f_{(h)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.51)$$

It follows from (4.21), (4.31), (4.41) and (4.42) that

$$f_{(h)} = q^{kl} a \Gamma (l + a) Y^{-l-a} + q^{kl} Q (Y, t) + q^{kl} E,$$

when $\nu = \nu_0$. Combining this result with (4.51), we find

$$f = q^{kl} a \Gamma (l + a) S_{p,q} Y^{-l-a} + \rho = \phi_{p,q} + \rho, \quad . \; . \; . \; . \; . \; . \; . \quad (4.52)$$

where

$$\rho = \rho_1 + \rho_2 + \rho_3, \quad \rho_1 = -1, \quad \rho_2 = q^{kl} \sum_{h=0}^{q-1} e_q (ph^k) E, \quad . \; . \; . \; . \quad (4.53)$$

$$\rho_3 = q^{kl} \sum_{h=0}^{q-1} e_q (ph^k) Q \left(Y, \frac{h}{q} \right). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.54)$$

Now

$$|\rho_2| \leq q^{kl} \sum_{h=0}^{q-1} |E| < \frac{B_\delta}{n} q^{kl+1} < B_\delta q^{kl}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.55)$$

Further,

$$Q \left(Y, \frac{h}{q} \right) = \sum_{j, m} c_{k, l, j, m} Y^j \left(\frac{h}{q} \right)^m,$$

where j, m and the coefficients have upper bounds B_δ; and

$$\begin{aligned} |\rho_3| &= q^{kl} \left| \sum_{h=0}^{q-1} e_q (ph^k) \sum_{j, m} c_{k, l, j, m} Y^j \left(\frac{h}{q} \right)^m \right| \\ &= q^{kl} \left| \sum_{j, m} c_{k, l, j, m} Y^j q^{-m} \sum_{h=0}^{q-1} h^m e_q (ph^k) \right| \\ &< B_{\epsilon, \delta} q^{kl+\kappa+\epsilon} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.56) \end{aligned}$$

by Lemma 6.

Lemma 5 follows from (4.52) to (4.56).

4.6. *Lemma* 11.—*If* $s \geq s_1$, *we have*

$$I_1 = \sum_{M} \int_{M} |\Pi f_i - \Pi \phi_i| \, d\theta < D n^{\Lambda l + sa - 1 - c}, \quad . \; . \; . \; . \; . \; . \; . \quad (4.61)$$

and, if $s \geq s_2$,

$$J_1 = \sum_{M} \int_{M} |\Pi f_i - \Pi \phi_i|^2 \, d\theta < D n^{\Lambda l + sa - 1 - c}, \quad . \; . \; . \; . \; . \; . \; . \quad (4.62)$$

provided that ε *is chosen sufficiently small in each case.*

Let us write

$$f_i = \phi_i + \Phi_i.$$

Then, if we put $\lambda_i l$ for l in Lemma 5, we have

$$|\Phi_i| = |f_i - \phi_{p, q, i}| < D_\epsilon q^{\lambda_i lk + \kappa + \epsilon} < D_\epsilon n^{\lambda_i l + a\kappa + \epsilon}$$

on $M_{p, q}$. Also, by (3.14), since $n|y| \geq 1$ on Γ,

$$|\phi_i| < D q^{-a} |y|^{-\lambda_i l - a} < D n^{\lambda_i l} q^{-a} |y|^{-a}.$$

We have

$$\Pi f_i - \Pi \phi_i = \Pi (\phi_i + \Phi_i) - \Pi \phi_i ;$$

and if this latter expression be expanded by multiplication, we see that it consists of $2^s - 1$ terms. Let us consider a term in which r of the factors are of the type ϕ; then the other s-r factors are of the type Φ; every suffix i appears once only and the absolute value of the term is less than

$$D_\epsilon n^{\Delta l + sa\kappa + s\epsilon} (n^\kappa q |y|)^{-ar}.$$

Then, since the term $\Pi \phi_i$ does not appear, we have on $M_{p,q}$

$$|\Pi f_i - \Pi \phi_i| < D_\epsilon n^{\Delta l + sa\kappa + s\epsilon} \sum_{r=0}^{s-1} (n^\kappa q |y|)^{-ar}$$
$$< D_\epsilon n^{\Delta l + sa\kappa + s\epsilon} (1 + (n^\kappa q |y|)^{-a(s-1)}).$$

Hence,

$$I_1 = \sum_M \int_M |\Pi f_i - \Pi \phi_i| d\theta$$
$$< D_\epsilon n^{\Delta l + s\epsilon} \left(n^{sa\kappa} \sum_M \int_M d\theta + n^{a\kappa} \sum_M q^{-a(s-1)} \int_M |y|^{-a(s-1)} d\theta \right).$$

Now, since $s \geq s_1 > 2k + 1$,

$$\int_M |y|^{-a(s-1)} d\theta < \int_{-\infty}^{+\infty} \frac{d\theta}{\left(\frac{1}{n^2} + \theta^2\right)^{\frac{1}{2}a(s-1)}} = n^{sa-a-1} \int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{1}{2}a(s-1)}} = Cn^{sa-a-1};$$

and so

$$\sum_M q^{-a(s-1)} \int_M |y|^{-a(s-1)} d\theta < Cn^{sa-a-1} \sum_{q \leq n^b} q^{1-a(s-1)} < Cn^{sa-a-1}.$$

Also,

$$\sum_M \int_M d\theta \leq \sum_{q \leq n^b} \sum_p \frac{2\pi}{qn^{1-b}} < 2\pi \sum_{q \leq n^b} n^{b-1} = An^{2b-1} < An^{2a-1}.$$

Hence, since $s \geq s_1 \geq 2K + 1$, we have

$$I_1 = D_\epsilon n^{\Delta l + s\epsilon} (n^{sa\kappa + 2a - 1} + n^{a\kappa + sa - a - 1}) \leq D_\epsilon n^{\Delta l + sa - 1 - \frac{a}{K} + s\epsilon}.$$

Then if we choose $\epsilon < (a/2sK)$, the first part of the lemma is proved.

Similarly, on $M_{p,q}$,

$$|\Pi f_i - \Pi \phi_i|^2 < D_\epsilon n^{2\Delta l + 2sa\kappa + 2s\epsilon} (1 + (n^\kappa q |y|)^{-2a(s-1)});$$

and, if $s \geq s_2$, we have

$$J_1 < Dn^{2\Delta l + 2sa - 1 - c}.$$

The minor arcs.

5.1. *Lemma* 12.—*On a minor arc* $\mathfrak{m}_{p,q}$, *we have*

$$|f(x)| < B_\epsilon n^{l + a - \frac{b}{K} + \epsilon}.$$

Let us take

$$\varepsilon' = \frac{\varepsilon}{kl+k+2}, \quad \omega = [n^{a+\epsilon'}],$$

and write

$$f(x) = \sum_{h=1}^{\omega} h^{lk} x^{h^k} + \sum_{\omega+1}^{\infty} h^{lk} x^{h^k} = f' + f''. \quad (5.11)$$

We can find a number $n' = B_{\epsilon'}$ such that, when $n > n'$, we have

$$ln < \omega^k = [n^{a+\epsilon'}]^k.$$

Then, for $n \leq n'$ and x on Γ,

$$|f''| \leq \sum_{h=\omega+1}^{\infty} h^{lk} e^{-h^k/n} \leq \sum_{h=\omega+1}^{\infty} h^{lk} e^{-h^k/n'} = B_{\epsilon'}.$$

If $n > n'$ and $u \geq \omega$, we have $ln < u^k$, and so

$$\frac{d}{du}(u^{lk} e^{-u^k/n}) = \left(l - \frac{u^k}{n}\right) ku^{lk-1} e^{-u^k/n} < 0.$$

Then,

$$f'' \leq \sum_{h=\omega+1}^{\infty} h^{lk} e^{-h^k/n} = (\omega+1)^{lk} e^{-(\omega+1)^k/n} + \sum_{h=\omega+2}^{\infty} h^{lk} e^{-h^k/n}.$$

We have

$$(\omega+1)^{lk} e^{-(\omega+1)^k/n} < B_{\epsilon'},$$

and

$$\sum_{h=\omega+2}^{\infty} h^{lk} e^{-h^k/n} < \int_{\omega+1}^{\infty} u^{lk} e^{-u^k/n} du < \int_{n^{a+\epsilon'}}^{\infty} u^{lk} e^{-u^k/n} du$$

$$= n^{l+a} \int_{n^{\epsilon'}}^{\infty} v^{lk} e^{-v^k} dv < n^{l+a} e^{-\frac{1}{2}n^{k\epsilon'}} \int_{n^{\epsilon'}}^{\infty} v^{lk} e^{-\frac{1}{2}v^k} dv$$

$$< n^{l+a} e^{-\frac{1}{2}n^{k\epsilon'}} \int_0^{\infty} v^{lk} e^{-\frac{1}{2}v^k} dv < B_{\epsilon'}.$$

Then, for all n, and all x on Γ, we have

$$|f''| < B_{\epsilon'} = B_{\epsilon}. \quad (5.12)$$

Let us now write

$$T_0 = 0, \quad T_m = \sum_{h=1}^{[ma]} e_q(ph^k), \quad (m > 0).$$

Then, if $n^b < q \leq n^{1-b}$, and $m \leq \omega^k$, it is known* that

$$|T_m| < A_{k,\epsilon'} n^{a-\frac{b}{K}+2\epsilon'}.$$

On $\mathfrak{m}_{p,q}$, since $q > n^b$, we have also

$$|1 - X| = |1 - e^{-y}| < A|y| < A\left(\frac{1}{n} + |\theta|\right) < A\left(\frac{1}{n} + \frac{1}{q[n^{1-b}]}\right) < \frac{A}{n};$$

* LANDAU, p. 262, bottom line.

and hence, since $|X| < 1$,

$$|m^l X^m - (m+1)^l X^{m+1}| \leq |m^l - (m+1)^l X| \leq m^l |1 - X| + Bm^{l-1} |X|$$
$$< B\left(\frac{\omega^{kl}}{n} + \omega^{k(l-1)}\right).$$

We have then, if x lies on $\mathbf{m}_{p,q}$,

$$|f'| = \left| \sum_{h=1}^{\omega} h^{lk} e_q(ph^k) X^{h^k} \right| = \left| \sum_{m=1}^{\omega^k} m^l X^m (T_m - T_{m-1}) \right|$$
$$\leq \sum_{m=1}^{\omega^k} |T_m (m^l X^m - (m+1)^l X^{m+1})| + |T_{\omega^k} (\omega^k + 1)^l X^{\omega^k + 1}|$$
$$\leq B_\epsilon n^{a - \frac{b}{K} + 2\epsilon'} \left\{ \omega^k \left(\frac{\omega^{kl}}{n} + \omega^{k(l-1)} \right) + \omega^{kl} \right\}$$
$$\leq B_\epsilon n^{l + a - \frac{b}{K} + \epsilon'(kl + k + 2)} = B_\epsilon n^{l + a - \frac{b}{K} + \epsilon}. \qquad (5.13)$$

The lemma follows at once from (5.11) to (5.13).

5.2. *Lemma* 13.—*We have*

$$\int_\Gamma |f_1 f_2 f_3 f_4| \, d\psi < B_\epsilon n^{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) l + 2a + \epsilon}.$$

We have

$$f_1(x) f_2(x) = \sum_{m=1}^{\infty} r_{l,2}(m) x^m,$$

where

$$r_{l,2}(m) = \sum_{\substack{m_1^k + m_2^k = m \\ m_1 > 0,\, m_2 > 0}} m_1^{\lambda_1 lk} m_2^{\lambda_2 lk}$$
$$\leq m^{(\lambda_1 + \lambda_2) l} \Sigma 1 = m^{(\lambda_1 + \lambda_2) l} r_{0,2}(m).$$

Now, it is known that*

$$\sum_{\mu=1}^{m} r_{0,2}^2(\mu) < A_{k,\epsilon} m^{2a + \epsilon}.$$

Hence,

$$\sum_{\mu=1}^{m} r_{l,2}^2(\mu) \leq m^{2(\lambda_1 + \lambda_2) l} \sum_{\mu=1}^{m} r_{0,2}^2(\mu) < A_{k,\epsilon} m^{2(\lambda_1 + \lambda_2) l + 2a + \epsilon}.$$

Then, if $R = e^{-1/n}$,

$$\int_\Gamma |f_1(x) f_2(x)|^2 \, d\psi = \int_0^{2\pi} |f_1(Re^{i\psi}) f_2(Re^{i\psi})|^2 \, d\psi$$
$$= 2\pi \sum_{m=1}^{\infty} r_{l,2}^2(m) R^{2m}$$
$$= A(1 - R^2) \sum_{m=1}^{\infty} \left\{ \sum_{\mu=1}^{m} r_{l,2}^2(\mu) \right\} R^{2m}$$
$$\leq A_{k,\epsilon} (1 - R^2) \sum_{m=1}^{\infty} m^{2(\lambda_1 + \lambda_2) l + 2a + \epsilon} R^{2m}$$
$$< \frac{B_\epsilon}{(1 - R^2)^{2(\lambda_1 + \lambda_2) l + 2a +}} < B_\epsilon n^{2(\lambda_1 + \lambda_2) l + 2a + \epsilon}.$$

* LANDAU, Satz 262.

Similarly,

$$\int_{\Gamma} |f_3(x) f_4(x)|^2 \, d\psi < B_{\epsilon} n^{2(\lambda_3+\lambda_4)l+2a+\epsilon}.$$

Finally,

$$\left(\int_{\Gamma} |f_1 f_2 f_3 f_4| \, d\psi\right)^2 \leq \int_{\Gamma} |f_1 f_2|^2 \, d\psi \int_{\Gamma} |f_3 f_4|^2 \, d\psi$$
$$< B_{\epsilon} n^{2(\lambda_1+\lambda_2+\lambda_3+\lambda_4)l+4a+2\epsilon},$$

and this proves the lemma.

5.3. *Lemma* 14.—*If* $s \geq s_1$, *we have*

$$I_3 = \sum_{\mathbf{m}} \int_{\mathbf{m}} |\Pi f_i| \, d\theta < D n^{\Lambda l + sa - 1 - c}, \quad \ldots \ldots \ldots \quad (5.31)$$

and, if $s \geq s_2$,

$$J_3 = \sum_{\mathbf{m}} \int_{\mathbf{m}} |\Pi f_i|^2 \, d\theta < D n^{2\Lambda l + 2sa - 1 - c}, \quad \ldots \ldots \ldots \quad (5.32)$$

provided ε *and* δ *are chosen sufficiently small in each case.*

In Lemma 12, let us put $\lambda_i l$ for l. Then, we see that

$$|f_i| < D_{\epsilon} n^{\lambda_i l + a - \frac{b}{K} + \epsilon}$$

on $\mathbf{m}_{p,q}$. Hence, by Lemma 13,

$$I_3 < D_{\epsilon} n^{(\Lambda - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) l + (s-4)\left(a - \frac{b}{K} + \epsilon\right)} \int_{\Gamma} |f_1 f_2 f_3 f_4| \, d\psi$$
$$< D n^{\Lambda l + sa - 1 - \mu + (s-3)\epsilon},$$

where

$$\mu = (s-4)\frac{b}{K} + 2a - 1 \geq (s_1 - 4)\frac{b}{K} + 2a - 1 = \frac{a}{K} - \frac{\delta}{K}(s_1 - 4).$$

The first part of the lemma follows at once, if we take ε and δ sufficiently small. The second part may be proved in the same way.

Proof of Theorem 7.

6.1. *Lemma* 15.—*If* $s \geq s_1$, *then*

$$I_4 = \sum_{\mathbf{M}} \int_{\overline{\mathbf{M}}_{p,q}} |F_{p,q}| \, d\theta > D n^{\Lambda l + sa - 1 - b(sa-2)} < D n^{\Lambda l + sa - 1 - c}. \quad . \quad . \quad (6.11)$$

On $\overline{\mathbf{M}}_{p,q}$,

$$|y| = \left|\frac{1}{n} - i\theta\right| > |\theta| \geq \frac{\pi}{qn^{1-b}} = \theta_0 \text{ (say)}.$$

We have then, by (3.11) and (3.14), since $sa > 1$,

$$\int_{\overline{\mathbf{M}}_{p,q}} |F_{p,q}| \, d\theta < D q^{-sa} \int_{\overline{\mathbf{M}}_{p,q}} |y|^{-\Lambda l - sa} \, d\theta < D q^{-sa} \int_{\theta_0}^{\infty} \theta^{-\Lambda l - sa} \, d\theta$$
$$= D q^{\Lambda l - 1} n^{(\Lambda l + sa - 1)(1-b)}.$$

Hence,

$$I_4 < Dn^{(\Delta l+sa-1)(1-b)} \sum_{q \leq n^b} \sum_p q^{\Delta l-1} < Dn^{\Delta l+sa-1-b(sa-2)}.$$

Since $s \geq s_1 > 2k+2$, we have

$$sa - 2 > a,$$

and so

$$I_4 < Dn^{\Delta l+sa-1-c}.$$

6.2. To prove Theorem 7 we take ε and δ sufficiently small for the conditions of Lemmas 11 and 14 to be satisfied, $s \geq s_1$, and $\beta = b$. We have

$$\sigma_l(n) = r_l(n) - \rho_l(n)$$
$$= \frac{1}{2\pi i}\int_\Gamma \frac{\Pi f_i - F_1}{x^{n+1}}\,dx - \Delta_l n^{\Delta l+sa-1} S_2(n);$$

and so

$$|\sigma_l(n)| \leq A\int_\Gamma |\Pi f_i - F_1|\,d\psi + \Delta_l n^{\Delta l+sa-1}|S_2(n)|$$
$$\leq A\left\{\sum_{\mathrm{M}}\int_{\mathrm{M}} |\Pi f_i - \Pi\phi_i|\,d\theta + \sum_{\mathrm{M}}\int_{\mathrm{M}} |F_{p,q} - \Pi\phi_i|\,d\theta\right.$$
$$\left.+\sum_{\mathrm{m}}\int_{\mathrm{m}} |\Pi f_i|\,d\theta + \sum_{\mathrm{M}}\int_{\overline{\mathrm{M}}_{p,q}} |F_{p,q}|\,d\theta\right\} + \Delta_l n^{\Delta l+sa-1}|S_2(n)|$$
$$= A(I_1 + I_2 + I_3 + I_4) + I_5.$$

By (3.15), since $s \geq s_1 > 2k+1$,

$$I_5 < Dn^{\Delta l+sa-1-b(sa-2)} < Dn^{\Delta l+sa-1-c}.$$

Then Theorem 7 follows from (3.12), (4.61), (5.31) and (6.11).

Proof of Theorem 8.

7.1. We require certain further lemmas to enable us to prove this theorem. We now take $s \geq s_2$, $\beta = \frac{1}{4}a$, and ε and δ sufficiently small to satisfy the conditions of Lemmas 11 and 14; in addition, we take $\delta < \frac{1}{4}a$, so that $3\beta < b$.

Lemma 16.—*If $s \geq s_2$, we have*

$$J_5 = \int_0^{2\pi} |F_2|^2\,d\psi < Dn^{2\Delta l+2sa-1-c}.$$

If $s \geq s_1$, by (3.16), we have

$$\int_0^{2\pi} |F_2|^2\,d\psi < 2\pi\,\Delta_l \sum_{m=1}^{\infty} (S_2(m))^2\, m^{2\Delta l+2sa-2}\,|x|^{2m}$$
$$< D\nu^{-c}\sum_{m=1}^{\infty} m^{2\Delta l+2sa-2+\epsilon}\,|x|^{2m}$$
$$< D_\epsilon n^{2\Delta l+2sa-1-\beta c+\varepsilon}$$
$$< Dn^{2\Delta l+2sa-1-c},$$

by suitable choice of ε.

Lemma 17.—*If* $s \geq s_2$, *then*

$$\sum_{q>\nu} \sum_{p} \int_{\xi_{p,q}} |\Pi \phi_{p,q,i}|^2 \, d\theta < \mathrm{D} n^{2\Lambda l+2sa-1-c}.$$

We have, by (3.14),

$$\int_{\xi_{p,q}} |\Pi \phi_{p,q,i}|^2 \, d\theta < \mathrm{D} q^{-2sa} \int_{\xi_{p,q}} \frac{d\theta}{|y|^{2\Lambda l+2sa}} < \mathrm{D} q^{-2sa} \int_{-\infty}^{+\infty} \frac{d\theta}{\left(\frac{1}{n^2}+\theta^2\right)^{\Lambda l+sa}}$$

$$= \mathrm{D} n^{2\Lambda l+2sa-1} q^{-2sa}.$$

Hence,

$$\sum_{q>\nu} \sum_{p} \int_{\xi_{p,q}} |\Pi \phi_i|^2 \, d\theta < \mathrm{D} n^{2\Lambda l+2sa-1} \sum_{q>\nu} q^{1-2sa}$$

$$< \mathrm{D} \nu^{2-2sa} n^{2\Lambda l+2sa-1} < \mathrm{D} n^{2\Lambda l+2sa-1-c},$$

since $s \geq s_2 > k$, and so $2sa > 2$.

7.2. *Lemma* 18.—*If* $s \geq s_2$, *we have*

$$\mathrm{J}_2 + \mathrm{J}_4 = \sum_{\mathbf{M}} \int_{\mathbf{M}} |\mathrm{F}_1 - \Pi \phi_i|^2 \, d\theta + \sum_{\mathbf{m}} \int_{\mathbf{m}} |\mathrm{F}_1|^2 \, d\theta < \mathrm{D} n^{2\Lambda l+2sa-1-c}.$$

We write

$$\mathrm{U} = \sum_{\mathrm{Q} \leq \nu} \sum_{\mathrm{P}} |\mathrm{F}_{\mathrm{P},\mathrm{Q}}|^2,$$

where P bears the same relation to Q as p to q. Then we have, if $q \leq \nu$,

$$\mathrm{F}_1 - \mathrm{F}_{p,q} = \sum_{\mathrm{Q} \leq \nu}' \sum_{\mathrm{P}} \mathrm{F}_{\mathrm{P},\mathrm{Q}},$$

where Σ' denotes that the term for which $\mathrm{Q} = q$, $\mathrm{P} = p$ is omitted. This sum contains less than ν^2 terms. Hence,

$$|\mathrm{F}_1 - \mathrm{F}_{p,q}|^2 \leq \nu^2 \sum_{\mathrm{Q} \leq \nu}' \sum_{\mathrm{P}} |\mathrm{F}_{\mathrm{P},\mathrm{Q}}|^2 = \nu^2 (\mathrm{U} - |\mathrm{F}_{p,q}|^2),$$

and

$$|\mathrm{F}_1 - \Pi \phi_{p,q,i}|^2 \leq 2|\mathrm{F}_1 - \mathrm{F}_{p,q}|^2 + 2|\mathrm{F}_{p,q} - \Pi \phi_{p,q,i}|^2$$

$$\leq 2\nu^2 (\mathrm{U} - |\mathrm{F}_{p,q}|^2) + 2|\mathrm{F}_{p,q} - \Pi \phi_{p,q,i}|^2. \quad \ldots \ldots \quad (7.21)$$

We have also

$$|\mathrm{F}_1|^2 \leq \nu^2 \mathrm{U} < 2\nu^2 \mathrm{U}, \quad \ldots \ldots \quad (7.22)$$

and so, if $\nu < q \leq n^b$,

$$|\mathrm{F}_1 - \Pi \phi_{p,q,i}|^2 \leq 2\nu^2 \mathrm{U} + 2|\Pi \phi_{p,q,i}|^2. \quad \ldots \ldots \quad (7.23)$$

We use (7.21) if $q \leq \nu$, (7.23) if $\nu < q \leq n^b$, and (7.22) for the minor arcs, that is, when $q > n^b$. Then we have

$$\mathrm{J}_2 + \mathrm{J}_4 \leq 2\nu^2 \left(\sum_{\mathbf{M}} \int_{\mathbf{M}} \mathrm{U} \, d\theta + \sum_{\mathbf{m}} \int_{\mathbf{m}} \mathrm{U} \, d\theta - \sum_{q \leq \nu} \sum_{p} \int_{\mathbf{M}_{p,q}} |\mathrm{F}_{p,q}|^2 \, d\theta \right)$$

$$+ 2 \sum_{q \leq \nu} \sum_{p} \int_{\mathbf{M}} |\mathrm{F}_{p,q} - \Pi \phi_{p,q,i}|^2 d\theta + 2 \sum_{\nu < q \leq n^b} \sum_{p} \int_{\mathbf{M}} |\Pi \phi_{p,q,i}|^2 \, d\theta.$$

By Lemmas 2 and 17, the last two sums are less than

$$\mathrm{D}n^{2\Lambda l+2sa-1-c}.$$

For the other sums we have

$$\sum_{\mathrm{M}}\int_{\mathrm{M}}\mathrm{U}\,d\theta+\sum_{\mathrm{m}}\int_{\mathrm{m}}\mathrm{U}\,d\theta=\int_{\Gamma}\mathrm{U}\,d\theta=\sum_{q\leq\nu}\sum_{p}\int_{\Gamma}|\mathrm{F}_{p,q}|^2\,d\theta,$$

and

$$\sum_{q\leq\nu}\sum_{p}\left(\int_{\Gamma}|\mathrm{F}_{p,q}|^2\,d\theta-\int_{\mathrm{M}_{p,q}}|\mathrm{F}_{p,q}|^2\,d\theta\right)=\sum_{q\leq\nu}\sum_{p}\int_{\overline{\mathrm{M}}_{p,q}}|\mathrm{F}_{p,q}|^2\,d\theta.$$

By the method of Lemma 15, we can show that

$$\int_{\overline{\mathrm{M}}_{p,q}}|\mathrm{F}_{p,q}|^2\,d\theta<\mathrm{D}q^{2\Lambda l+1}n^{(2\Lambda l+2sa-1)(1-b)}.$$

Hence,

$$\mathrm{J}_2+\mathrm{J}_4<\mathrm{D}n^{2\Lambda l+2sa-1-c}+\mathrm{D}\nu^2n^{(2\Lambda l+2sa-1)(1-b)}\sum_{q\leq\nu}\sum_{p}q^{2\Lambda l-1}.$$

$$\leq\mathrm{D}n^{2\Lambda l+2sa-1}\left(n^{-c}+n^{\beta(2\Lambda l+3)-b(2\Lambda l+2sa-1)}\right).$$

Since $s\geq s_2>k$, we have $2sa-2>0$ and

$$-b(2\Lambda l+2sa-1)+\beta(2\Lambda l+3)<-b(2sa-2)<-c.$$

This completes the proof of the lemma.

7.3. We have

$$\sum_{m=1}^{\infty}\sigma_l(m)\,x^m=\Pi f_i(x)-\mathrm{F}(x)=\Pi f_i-\mathrm{F};$$

$$\sum_{m=1}^{\infty}(\sigma_l(m))^2\,|x|^{2m}=\frac{1}{2\pi}\int_0^{2\pi}|\Pi f_i-\mathrm{F}|^2\,d\psi;$$

and so

$$\sum_{m=1}^{n}(\sigma_l(m))^2\leq e^2\sum_{m=1}^{n}(\sigma_l(m))^2\,e^{-2m/n}<\mathrm{A}\int_{\Gamma}|\Pi f_i-\mathrm{F}|^2\,d\psi$$

$$\leq\mathrm{A}\left(\sum_{\mathrm{M}}\int_{\mathrm{M}}|\Pi f_i-\Pi\phi_i|^2\,d\theta+\sum_{\mathrm{M}}\int_{\mathrm{M}}|\mathrm{F}_1-\Pi\phi_i|^2\,d\theta\right.$$

$$\left.+\sum_{\mathrm{m}}\int_{\mathrm{m}}|\Pi f_i|^2\,d\theta+\sum_{\mathrm{m}}\int_{\mathrm{m}}|\mathrm{F}_1|^2\,d\theta+\int_0^{2\pi}|\mathrm{F}_2|^2\,d\psi\right)$$

$$=\mathrm{A}(\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4+\mathrm{J}_5).$$

Then Theorem 8 follows from Lemmas 11, 14, 16 and 18.

Proof of Theorems 4, 5 *and* 6.

8.1. Our "singular series" $\mathrm{S}(n)$ is the same as that introduced by HARDY and LITTLEWOOD. Our next two lemmas are due to these authors.

Lemma 19.*—*If* $s \geq s_1$, *then* $S(n) > A_{k,s} > C$.

Lemma 20.†—*If* (i) $k \neq 4$ and $s \geq s_2$, *or if* (ii) $k = 4$, $s \geq 15$, *and* $n \not\equiv 0$ (*mod* 16), *then* $S(n) > A_{k,s} > C$.

8.2. *Lemma* 21.—*If* $\gamma' > 0$, *there exists a positive integer* $L = L(k, \gamma', \lambda)$ *such that*‡

$$\tau = \tau(k, \lambda, L) = \frac{\Lambda^\Lambda}{\Pi \lambda_i^{\lambda_i}} \left(\frac{\Delta_L}{2\Delta_0}\right)^{1/L} > 1 - \gamma'.$$

Taking logarithms, we have

$$\left|\log \tau - \Lambda \log \Lambda + \sum_{i=1}^{s} \lambda_i \log \lambda_i - \frac{1}{L} \sum_{i=1}^{s} \log \Gamma(\lambda_i L + a) + \frac{1}{L} \log \Gamma(\Lambda L + sa)\right| < \frac{C}{L}.$$

If we now substitute Stirling's series for the logarithms of the Gamma-functions, this becomes

$$\left|\log \tau + \tfrac{1}{2}(s-1)\frac{\log L}{L}\right| < \frac{C}{L}.$$

Then we have

$$|\log \tau| < \frac{C \log L}{L} < \gamma',$$

if L is large enough ; that is, if $L = L(k, \lambda, \gamma')$. If $\log \tau$ is positive or zero, the lemma is obvious ; while if $\log \tau$ is negative, we have

$$\tau = e^{-|\log \tau|} > e^{-\gamma'} > 1 - \gamma'.$$

Lemma 22.—*If* (i) $S(m) > C$,

$$\text{(ii)} \quad |\sigma_0(m)| < \tfrac{1}{3}\rho_0(m), \qquad (8.21)$$

and

$$\text{(iii)} \quad |\sigma_L(m)| < \tfrac{1}{3}\rho_L(m), \qquad (8.22)$$

then $\overline{P}(m)$ *is defined and we have*

$$\overline{P}(m) > 1 - \gamma'.$$

Since $S(m) > C$, we have

$$\rho_0(m) = \Delta_0 m^{sa-1} S(m) > 0.$$

Then

$$r_0(m) = \rho_0(m) + \sigma_0(m) > \tfrac{2}{3}\rho_0(m) > 0\,;$$

hence there is at least one solution of

$$m = m_1^k + m_2^k + \dots + m_s^k$$

* LANDAU, Satz 325.

† LANDAU, Satz 326.

‡ We take the real, positive L-th root of $(\Delta_L/2\Delta_0)$, and the logarithms used in the proof are real.

in positive integers, and $\overline{\mathrm{P}}(m)$ is defined. Also,

$$r_0(m) = \rho_0(m) + \sigma_0(m) < \tfrac{4}{3}\rho_0(m); \quad r_{\mathrm{L}}(m) = \rho_{\mathrm{L}}(m) + \sigma_{\mathrm{L}}(m) > \tfrac{2}{3}\rho_{\mathrm{L}}(m);$$

and so, by (2.22),

$$\overline{\mathrm{P}}(m) \geq \frac{\Lambda^{\Lambda}}{(\Pi\,\lambda_i^{\Lambda_i})\,n^{\Lambda}}\left(\frac{\tau_{\mathrm{L}}(m)}{\tau_0(m)}\right)^{1/\mathrm{L}} > \frac{\Lambda^{\Lambda}}{(\Pi\,\lambda_i^{\Lambda_i})\,n^{\Lambda}}\left(\frac{\rho_{\mathrm{L}}(m)}{2\rho_0(m)}\right)^{1/\mathrm{L}} = \tau > 1 - \gamma'.$$

8.3. *Proof of Theorem* 4.—If $s \geq s_1$, by Lemma 19,

$$\mathrm{S}(n) > \mathrm{A}_{k,s} = \mathrm{C}.$$

Then, for all $l \geq 0$,

$$\rho_l(n) = \Delta_l n^{\Lambda l + sa - 1}\,\mathrm{S}(n) > \mathrm{D}n^{\Lambda l + sa - 1}.$$

By Theorem 7,

$$\sigma_l(n) < \mathrm{D}n^{\Lambda l + sa - 1 - c} < \mathrm{D}n^{-c}\rho_l(n).$$

Then for every l there exists a number $\mathrm{N}_l = \mathrm{N}(k, l, \lambda)$, such that $n > \mathrm{N}_l$ implies that

$$\sigma_l(n) < \tfrac{1}{3}\rho_l(n).$$

If we take n_0 as the greater of the two numbers N_0, N_{L}, we have

$$\sigma_0(n) < \tfrac{1}{3}\rho_0(n), \; \sigma_{\mathrm{L}}(n) < \tfrac{1}{3}\rho_{\mathrm{L}}(n),$$

provided $n > n_0$. We see that

$$n_0 = n_0(k, \mathrm{L}, \lambda) = n_0(k, \gamma', \lambda).$$

Then Theorem 4 follows at once by means of Lemma 22.

8.4. *Proof of Theorem* 5.—If $k \neq 4$ and $s \geq s_2$, then

$$\mathrm{S}(m) > \mathrm{A}_{k,s} = \mathrm{C}.$$

We divide the positive integers into three classes M_1, M_2, M_3 as follows. m belongs to M_1 if (8.21) and (8.22) are both true; m belongs to M_2 if (8.21) is false, and to M_3 if (8.21) is true and (8.22) false. Then, by Lemma 28, if m belongs to M_1, $\overline{\mathrm{P}}(m)$ is defined and

$$\overline{\mathrm{P}}(m) > 1 - \gamma'.$$

We have only to prove that the number of integers belonging to M_2 or M_3 less than n is less than $\mathrm{C}_{\gamma'}\,n^{1-c}$.

Let $\mu(n)$ be the number of integers belonging to M_2 such that

$$\tfrac{1}{2}n \leq m < n.$$

If m belongs to M_2,

$$|\sigma_0(m)| \geq \tfrac{1}{3}\rho_0(m) = \tfrac{1}{3}\Delta_0\,\mathrm{S}(m)\,m^{sa-1} > Cm^{sa-1}.$$

Hence,

$$\sum_{\frac{1}{2}n \leq m < n} (\sigma_0(m))^2 \geq C(\tfrac{1}{2}n)^{2sa-2}\mu(n);$$

and so, by Theorem 8,

$$\mu(n) \leq Cn^{1-c}.$$

The total number of integers belonging to M_2 and less than n is then

$$\mu(n) + \mu\left(\frac{n}{2}\right) + \mu\left(\frac{n}{4}\right) + \dots \leq Cn^{1-c}(1 + (\tfrac{1}{2})^{1-c} + (\tfrac{1}{4})^{1-c} + \dots) = Cn^{1-c}.$$

Similarly, the number of integers belonging to M_3 and less than n is less than $C_L n^{1-c}$. Then, the number of values of m less than n for which the hypotheses of Lemma 22 are not fulfilled is less than

$$Cn^{1-c} + C_L n^{1-c} = C_{\gamma'} n^{1-c}.$$

8.5. *Proof of Theorem* 6.—Let us call exceptional an integer m such that the hypotheses of Lemma 22 are not fulfilled. If $k = 4$, $s \geq 15$, and $m \not\equiv 0 \pmod{16}$, then

$$S(m) > A_{k,s} = C.$$

We can divide the integers which are not multiples of 16 into three classes as in 8.4 and prove in the same way that, for almost all of these, $\overline{P}(m)$ is defined and

$$\overline{P}(m) > 1 - \gamma'.$$

Also, the number of exceptional integers not multiples of 16 and less than n is, as before, less than $C_{\gamma'} n^{1-c}$.

If $m \equiv 0 \pmod{16}$, let us write

$$m = 16^t m',$$

where $m' \not\equiv 0 \pmod{16}$. Then

$$m' = \frac{m}{16^t} < \frac{n}{16^t}.$$

If $\overline{P}(m')$ exists and $\overline{P}(m') > 1 - \gamma'$, then

$$m' = \sum_{i=1}^{s} m_i'^4,$$

and therefore

$$m = 16^t m' = \sum_{i=1}^{s} (2^t m_i')^4 = \sum_{i=1}^{s} m_i^4.$$

Also,

$$\overline{P}(m) \geq \overline{P}(m') > 1 - \gamma'.$$

Now the number of exceptional values of m', not multiples of 16 and less than $n/16^t$, is less than

$$C_{\gamma'}\left(\frac{n}{16^t}\right)^{1-c}.$$

Hence, the number of exceptional values of m less than n is less than

$$C_{\gamma'}\left(n^{1-c}+\left(\frac{n}{16}\right)^{1-c}+\left(\frac{n}{16^2}\right)^{1-c}+\dots\right)=C_{\gamma}n^{1-c}.$$

Proof of Theorems 1, 2 *and* 3.

9.1. *Lemma* 23.—*If* $\mu_1, \mu_2, \dots \mu_t$ *are* t *positive numbers such that*

$$\mu_1+\mu_2+\dots+\mu_t=t,$$

and

$$\mu_1\,\mu_2\dots\mu_t=G^t,$$

then

$$|\mu_i-1|\leq 2t\,(1-G)^{\frac{1}{2}},\qquad (i=1,2,\dots t).$$

Every μ_i is positive; the arithmetic mean is unity and the geometric mean is G. Hence $0<G\leq 1$. But, if $G=1$, we know that

$$\mu_1=\mu_2=\dots=\mu_t=1.$$

If $G\neq 1$, then the numbers μ_i are not all equal. Let us write $\mu_i-1=\varepsilon$; and suppose μ_1 the greatest, μ_2 the least, of $\mu_1, \mu_2, \dots \mu_t$. Then

$$\mu_2<G<1<\mu_1,$$

and so

$$\varepsilon_2<0<\varepsilon_1.$$

The lemma is true if

$$\varepsilon_1\leq 2t\,(1-G)^{\frac{1}{2}},\qquad -\varepsilon_2\leq 2t\,(1-G)^{\frac{1}{2}}.$$

Let us suppose that

$$\varepsilon_1>2t\,(1-G)^{\frac{1}{2}}.$$

Then

$$-\varepsilon_2>2\,(1-G)^{\frac{1}{2}};$$

for otherwise

$$\varepsilon_1=-\sum_{i=2}^{t}\varepsilon_i\leq 2\,(t-1)\,(1-G)^{\frac{1}{2}}<2t\,(1-G)^{\frac{1}{2}}.$$

Now let us replace μ_1 by G, μ_2 by $(\mu_1\,\mu_2)/G$. Then the geometric mean is still G, but the arithmetic mean is now

$$\frac{1}{t}\left(G+\frac{\mu_1\mu_2}{G}+\mu_3+\dots+\mu_t\right)=1+\frac{1}{t}\left(G+\frac{\mu_1\mu_2}{G}-\mu_1-\mu_2\right)$$

$$=1-\frac{(\mu_1-G)\,(G-\mu_2)}{Gt}\Big).$$

This must be greater than or equal to G. Hence,

$$(\mu_1-G)\,(G-\mu_2)\leq G\,t\,(1-G).$$

But

$$\mu_1-G>\mu_1-1=\varepsilon_1>2t\,(1-G)^{\frac{1}{2}};$$

and

$$G-\mu_2=-\varepsilon_2-(1-G)>2\,(1-G)^{\frac{1}{2}}-(1-G)>0.$$

Combining the last three inequalities, we have

$$4t(1 - G)(1 - (1 - G)^{\frac{1}{2}}) < Gt(1 - G),$$

from which we deduce

$$8G + G^2 < 0.$$

This is impossible, and so

$$\varepsilon_1 \leq 2t(1 - G)^{\frac{1}{2}}.$$

In the same way, the hypothesis

$$-\varepsilon_2 > 2t(1 - G)^{\frac{1}{2}}$$

leads to a contradiction. Hence, the lemma is true.

Lemma 24.—*If* $\gamma < 2\Lambda$ *and*

$$P(n) > \left(1 - \left(\frac{\gamma}{2\Lambda}\right)^2\right)^\Lambda,$$

then

$$1 - \gamma < \alpha_i < 1 + \gamma, \qquad (i = 1, 2, \ldots s).$$

Consider the Λ positive numbers in (2.11). Their sum is Λ and their product is $P(n)$. Hence,

$$G = (P(n))^{1/\Lambda} > 1 - \left(\frac{\gamma}{2\Lambda}\right)^2,$$

and by Lemma 23,

$$|\alpha_i - 1| \leq 2\Lambda(1 - G)^{\frac{1}{2}} < \gamma.$$

9.2. It is obviously sufficient to prove Theorems 1, 2 and 3 for the case $\gamma < 2\Lambda$. We put

$$\gamma' = 1 - \left(1 - \left(\frac{\gamma}{2\Lambda}\right)^2\right)^\Lambda$$

in Theorems 4, 5 and 6.

Now if $\overline{P}(m)$ is defined and is greater than $1 - \gamma'$ for a particular set of values of m, k, s, λ, there is at least one solution of the equation

$$m = m_1^k + m_2^k + \ldots + m_s^k$$

in positive integers. In addition, if we take the particular solution for which

$$P(m) = \overline{P}(m),$$

we have

$$P(m) > 1 - \gamma' = \left(1 - \left(\frac{\gamma}{2\Lambda}\right)^2\right)^\Lambda,$$

and so

$$1 - \gamma < \alpha_i < 1 + \gamma, \qquad (i = 1, 2, \ldots s),$$

and

$$(1 - \gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i^k}{n} < (1 + \gamma)\frac{\lambda_i}{\Lambda}.$$

We see then that Theorems 1, 2 and 3 follow from Theorems 4, 5 and 6 respectively.

Summary.

The question with which this paper is concerned is the following. k, s, n are positive integers, $k \geq 3$, and γ, λ_1, λ_2, ... λ_s are any positive numbers; also

$$\Lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_s.$$

For what values of n has the equation

$$m_1^k + m_2^k + \ldots + m_s^k = n \quad \ldots \ldots \ldots \quad (1)$$

a solution in positive integers m_1, m_2, ... m_s satisfying the conditions

$$(1-\gamma)\frac{\lambda_i}{\Lambda} < \frac{m_i^k}{n} < (1+\gamma)\frac{\lambda_i}{\Lambda}, \quad (i = 1, 2, \ldots s)\ ? \quad \ldots \ldots \quad (2)$$

The answers found to this question are :—

(a) If $s \geq (k-2)\,2^{k-1} + 5$, for all n greater than a certain n_0 depending on k, s, γ, λ_1, λ_2, ... λ_s.

(b) If $k \neq 4$ and $s \geq (\frac{1}{2}k - 1)\,2^{k-1} + 3$, or if $k = 4$ and $s \geq 15$, for almost all n. The number of values of n less than N for which there is no solution is $O(N^{1-c})$, where $c = c(k, s) > 0$.

The fundamental idea of the method of proof is that of weighting the various solutions of (1), so that those satisfying (2) have predominantly large weights. Thus we construct the function

$$r_l(n) = \sum_{\substack{m_1^k + \ldots + m_s^k = n, \\ m_i > 0.}} (m_1^{\lambda_1} m_2^{\lambda_2} \ldots m_s^{\lambda_s})^{kl},$$

where l is a large positive integer. We then find the asymptotic value of this for a fixed l and large n by an adaptation of the well-known HARDY-LITTLEWOOD method for WARING's Problem.
